Unsupervised Learning with Gaussian Processes

Neil D. Lawrence

GPSS
17th September 2014

Outline

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Motivation for Non-Linear Dimensionality Reduction

USPS Data Set Handwritten Digit

- 3648 Dimensions
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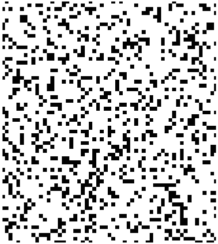

Motivation for Non-Linear Dimensionality Reduction

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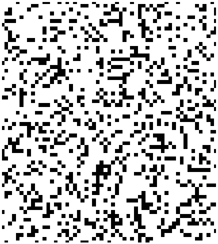

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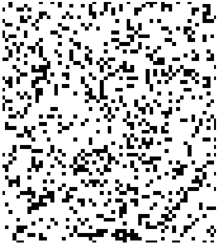

Simple Model of Digit

Rotate a 'Prototype'

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MATLAB Demo

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demDigitsManifold([1 2], 'all')
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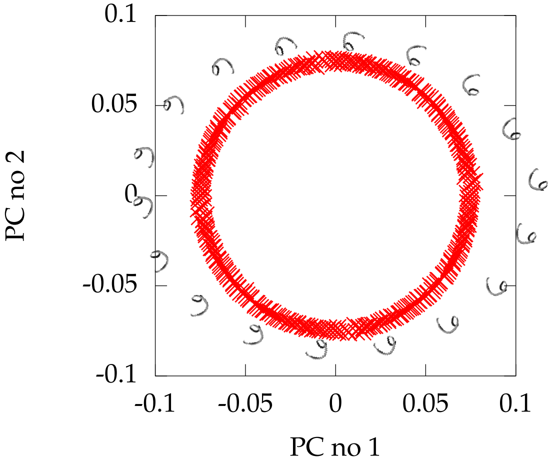

MATLAB Demo

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demDigitsManifold([1 2], 'sixnine')
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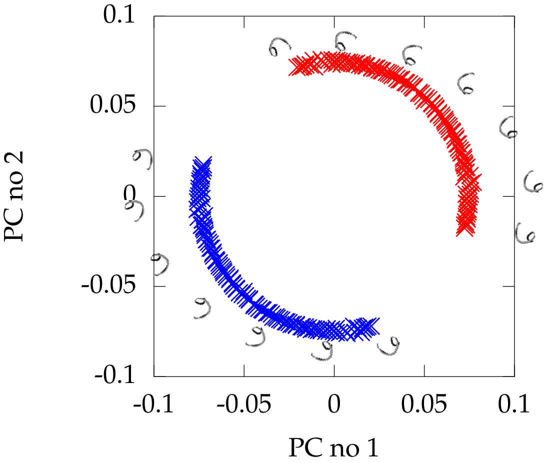


Low Dimensional Manifolds

Pure Rotation is too Simple

- In practice the data may undergo several distortions.
  - *e.g.* digits undergo 'thinning', translation and rotation.
- For data with 'structure':
  - we expect fewer distortions than dimensions;
  - we therefore expect the data to live on a lower dimensional manifold.
- Conclusion: deal with high dimensional data by looking for lower dimensional non-linear embedding.

Outline

Notation

q— dimension of latent/embedded space
p— dimension of data space
n— number of data points

$$\text{data, } \mathbf{Y} = [\mathbf{y}_{1,:}, \ldots, \mathbf{y}_{n,:}]^\top = \left[\mathbf{y}_{:,1}, \ldots, \mathbf{y}_{:,p}\right] \in \Re^{n \times p}$$

$$\textit{centred} \text{ data, } \hat{\mathbf{Y}} = [\hat{\mathbf{y}}_{1,:}, \ldots, \hat{\mathbf{y}}_{n,:}]^\top = \left[\hat{\mathbf{y}}_{:,1}, \ldots, \hat{\mathbf{y}}_{:,p}\right] \in \Re^{n \times p},$$

$$\hat{\mathbf{y}}_{i,:} = \mathbf{y}_{i,:} - \boldsymbol{\mu}$$

$$\text{latent variables, } \mathbf{X} = [\mathbf{x}_{1,:}, \ldots, \mathbf{x}_{n,:}]^\top = \left[\mathbf{x}_{:,1}, \ldots, \mathbf{x}_{:,q}\right] \in \Re^{n \times q}$$

$$\text{mapping matrix, } \mathbf{W} \in \Re^{p \times q}$$

$\mathbf{a}_{i,:}$ is a vector from the ith row of a given matrix $\mathbf{A}$
$\mathbf{a}_{:,j}$ is a vector from the jth row of a given matrix $\mathbf{A}$

Reading Notation

$\mathbf{X}$ and $\mathbf{Y}$ are *design matrices*

- Data covariance given by $\frac{1}{n}\hat{\mathbf{Y}}^\top\hat{\mathbf{Y}}$

$$\mathrm{cov}\left(\mathbf{Y}\right) = \frac{1}{n}\sum_{i=1}^{n}\hat{\mathbf{y}}_{i,:}\hat{\mathbf{y}}_{i,:}^\top = \frac{1}{n}\hat{\mathbf{Y}}^\top\hat{\mathbf{Y}} = \mathbf{S}.$$

- Inner product matrix given by $\mathbf{Y}\mathbf{Y}^\top$

$$\mathbf{K} = \left(k_{i,j}\right)_{i,j}, \qquad k_{i,j} = \mathbf{y}_{i,:}^\top\mathbf{y}_{j,:}$$

Linear Dimensionality Reduction

- Find a lower dimensional plane embedded in a higher dimensional space.
- The plane is described by the matrix $\mathbf{W} \in \Re^{p\times q}$.

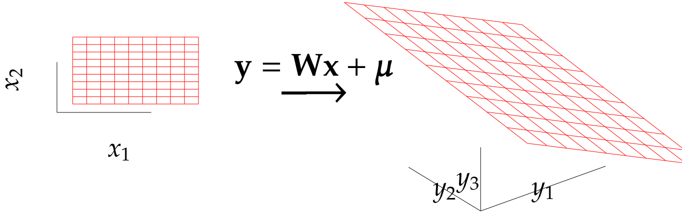


Figure : Mapping a two dimensional plane to a higher dimensional space in a linear way. Data are generated by corrupting points on the plane with noise.

Linear Dimensionality Reduction

Linear Latent Variable Model

- Represent data, $\mathbf{Y}$, with a lower dimensional set of latent variables $\mathbf{X}$.
- Assume a linear relationship of the form

$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:},$$

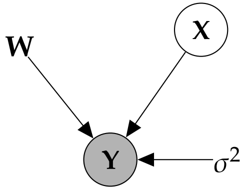
where

$$\boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right).$$

Linear Latent Variable Model

Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.



$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right)=\prod_{i=1}^{n}\mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^{2}\mathbf{I}\right)$$

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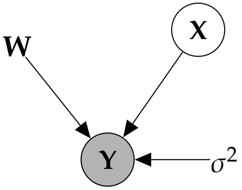

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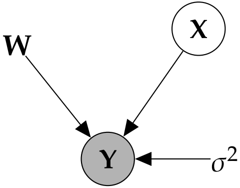

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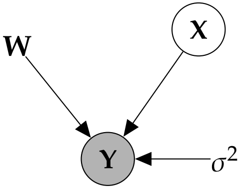

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$$\mathbf{y}_{i,:} = \mathbf{W}\mathbf{x}_{i,:} + \boldsymbol{\epsilon}_{i,:}, \quad \mathbf{x}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \quad \boldsymbol{\epsilon}_{i,:} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$$

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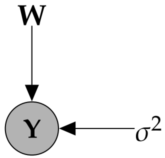
Linear Latent Variable Model II

Probabilistic PCA Max. Likelihood Soln (Tipping and Bishop, 1999)



$$p\left(\mathbf{Y}|\mathbf{W}\right) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{0}, \mathbf{W}\mathbf{W}^{\top} + \sigma^2\mathbf{I}\right)$$

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Difficulty for Probabilistic Approaches

- Propagate a probability distribution through a non-linear mapping.
- Normalisation of distribution becomes intractable.

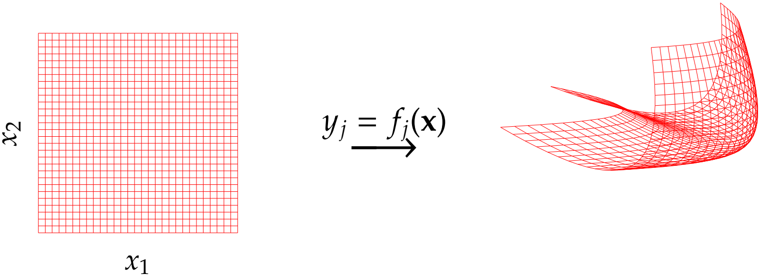


Figure : A three dimensional manifold formed by mapping from a two dimensional space to a three dimensional space.

Difficulty for Probabilistic Approaches

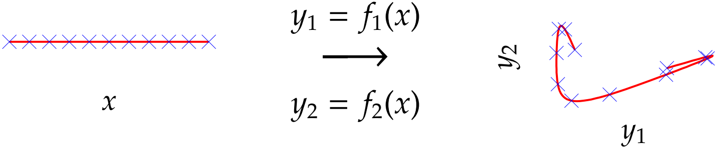


Figure : A string in two dimensions, formed by mapping from one dimension, x, line to a two dimensional space, $[y_1,\ y_2]$ using nonlinear functions $f_1(\cdot)$ and $f_2(\cdot)$.

Difficulty for Probabilistic Approaches

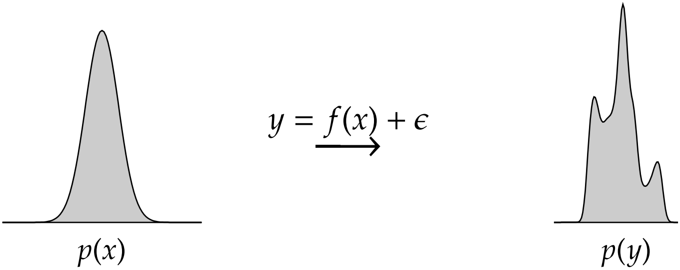
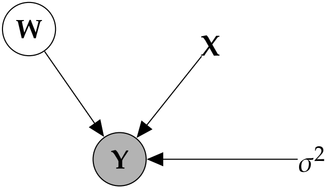


Figure : A Gaussian distribution propagated through a non-linear mapping. $y_i = f(x_i) + \epsilon_i$. $\epsilon \sim \mathcal{N}\left(0, 0.2^2\right)$ and $f(\cdot)$ uses RBF basis, 100 centres between -4 and 4 and $\ell = 0.1$. New distribution over y (right) is multimodal and difficult to normalize.

Linear Latent Variable Model III

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.



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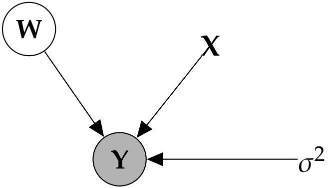

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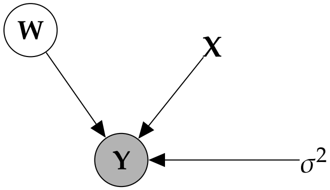

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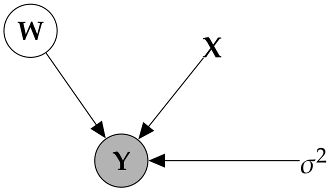

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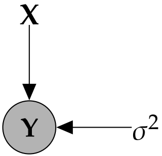
Dual Probabilistic PCA Max. Likelihood Soln (Lawrence, 2004, 2005)



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where $\mathbf{R}$ is an arbitrary rotation matrix.

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Equivalence of Formulations

The Eigenvalue Problems are equivalent

- Solution for Probabilistic PCA (solves for the mapping)

$$\mathbf{Y}^\top \mathbf{Y} \mathbf{U}_q = \mathbf{U}_q \boldsymbol{\Lambda}_q \qquad \mathbf{W} = \mathbf{U}_q \mathbf{L} \mathbf{R}^\top$$

- Solution for Dual Probabilistic PCA (solves for the latent positions)

$$\mathbf{Y} \mathbf{Y}^\top \mathbf{U}'_q = \mathbf{U}'_q \boldsymbol{\Lambda}_q \qquad \mathbf{X} = \mathbf{U}'_q \mathbf{L} \mathbf{R}^\top$$

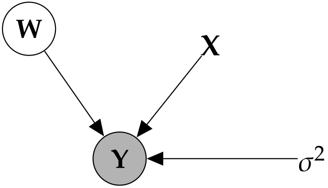
- Equivalence is from

$$\mathbf{U}_q = \mathbf{Y}^\top \mathbf{U}'_q \boldsymbol{\Lambda}_q^{-\frac{1}{2}}$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Novel** Latent variable approach:
  - Define Gaussian prior over *parameteters*, $\mathbf{W}$.
  - Integrate out *parameters*.



$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:}, \sigma^2\mathbf{I}\right)$$

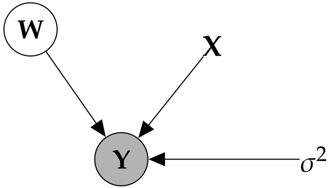
$$p\left(\mathbf{W}\right) = \prod_{i=1}^{p} \mathcal{N}\left(\mathbf{w}_{i,:}|\mathbf{0}, \mathbf{I}\right)$$

$$p\left(\mathbf{Y}|\mathbf{X}\right) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{X}\mathbf{X}^\top + \sigma^2\mathbf{I}\right)$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...

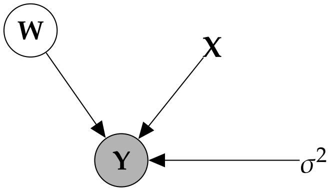


$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{X}\mathbf{X}^\top + \sigma^2\mathbf{I}\right)$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.



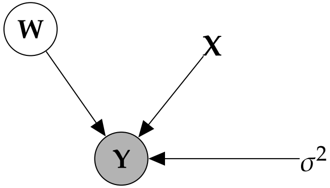
$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$\mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.
  - We recognise it as the 'linear kernel'.



$$p\left(\mathbf{Y}|\mathbf{X}\right) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

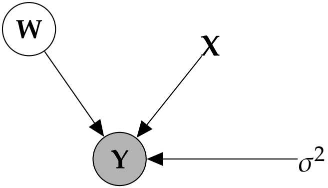
$$\mathbf{K} = \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}$$

This is a product of Gaussian processes with linear kernels.

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.
  - We recognise it as the ‘linear kernel’.
  - We call this the Gaussian Process Latent Variable model (GP-LVM).



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$\mathbf{K} = ?$$

Replace linear kernel with non-linear kernel for non-linear model.

Exponentiated Quadratic (EQ) Covariance

- The EQ covariance has the form $k_{i,j} = k\left(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}\right)$, where

$$k\left(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}\right) = \alpha \exp\left(-\frac{\left\|\mathbf{x}_{i,:} - \mathbf{x}_{j,:}\right\|_2^2}{2\ell^2}\right).$$

- No longer possible to optimise wrt $\mathbf{X}$ via an eigenvalue problem.
- Instead find gradients with respect to $\mathbf{X}, \alpha, \ell$ and σ^2 and optimise using conjugate gradients.

Applications

Style Based Inverse Kinematics

- Facilitating animation through modeling human motion (Grochow et al., 2004)

Tracking

- Tracking using human motion models (Urtasun et al., 2005, 2006)

Assisted Animation

- Generalizing drawings for animation (Baxter and Anjyo, 2006)

Shape Models

- Inferring shape (e.g. pose from silhouette). (Ek et al., 2008b,a; Priacuriu and Reid, 2011a,b)

Stick Man

Generalization with less Data than Dimensions

- Powerful uncertainly handling of GPs leads to surprising properties.
- Non-linear models can be used where there are fewer data points than dimensions *without overfitting*.
- Example: Modelling a stick man in 102 dimensions with 55 data points!

`demStick1`

Figure : The latent space for the stick man motion capture data.

Stick Man II

`demStick1`

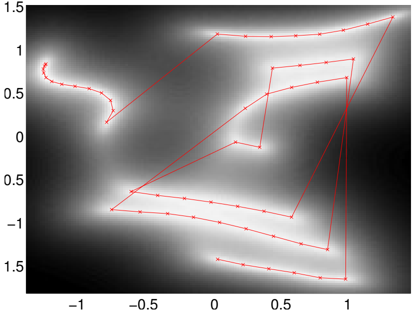


Figure : The latent space for the stick man motion capture data.

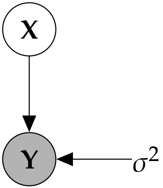
Selecting Data Dimensionality

- GP-LVM Provides probabilistic non-linear dimensionality reduction.
- How to select the dimensionality?
- Need to estimate marginal likelihood.
- In standard GP-LVM it increases with increasing q.

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.

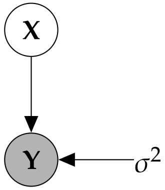


$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.
- Apply standard latent variable approach:
  - Define Gaussian prior over *latent space*, **X**.



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.
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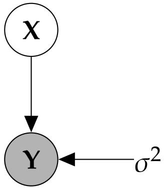
  - Define Gaussian prior over *latent space*, $\mathbf{X}$.
  - Integrate out *latent variables*.



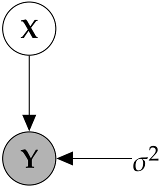
$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \alpha_i^{-2}\mathbf{I}\right)$$

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.
- Apply standard latent variable approach:
  - Define Gaussian prior over *latent space*, $\mathbf{X}$.
  - Integrate out *latent variables*.
  - Unfortunately integration is intractable.



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \alpha_i^{-2}\mathbf{I}\right)$$

$$p(\mathbf{Y}|\boldsymbol{\alpha}) = ??$$

Standard Variational Approach Fails

- Standard variational bound has the form:

$$\mathcal{L} = \langle \log p(\mathbf{y}|\mathbf{X}) \rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)$$

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- Requires expectation of $\log p(\mathbf{y}|\mathbf{X})$ under $q(\mathbf{X})$.

$$\log p(\mathbf{y}|\mathbf{X}) = -\frac{1}{2}\mathbf{y}^\top \left(\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right)^{-1} \mathbf{y} - \frac{1}{2}\log\left|\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right| - \frac{n}{2}\log 2\pi$$

Standard Variational Approach Fails

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- Extremely difficult to compute because $\mathbf{K}_{\mathbf{f},\mathbf{f}}$ is dependent on $\mathbf{X}$ and appears in the inverse.

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$p(\mathbf{y}) \geq \prod_{i=1}^{n} c_i \int \mathcal{N}\left(\mathbf{y} | \langle \mathbf{f} \rangle, \sigma^2 \mathbf{I}\right) p(\mathbf{u}) \mathrm{d}\mathbf{u}$$

Variational Bayesian GP-LVM

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$$p(\mathbf{y}|\mathbf{X}) \geq \prod_{i=1}^{n} c_i \int \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)p(\mathbf{u})\mathrm{d}\mathbf{u}$$

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$\int p(\mathbf{y}|\mathbf{X})p(\mathbf{X})\mathrm{d}\mathbf{X} \geq \int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle\mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}p(\mathbf{u})\mathrm{d}\mathbf{u}$$

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- Apply variational lower bound to the inner integral.

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- Apply variational lower bound to the inner integral.

$$\int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}$$

$$\geq \left\langle \sum_{i=1}^{n} \log c_i \right\rangle_{q(\mathbf{X})} + \left\langle \log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)\right\rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X})\,\|\,p(\mathbf{X})\right)$$

Variational Bayesian GP-LVM

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$$\int p(\mathbf{y}|\mathbf{X})p(\mathbf{X})\mathrm{d}\mathbf{X} \geq \int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X} p(\mathbf{u})\mathrm{d}\mathbf{u}$$

- Apply variational lower bound to the inner integral.

$$\begin{aligned}\int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X} \\ \geq \left\langle \sum_{i=1}^{n} \log c_i \right\rangle_{q(\mathbf{X})} \\ + \left\langle \log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)\right\rangle_{q(\mathbf{X})} \\ + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)\end{aligned}$$

- Which is analytically tractable for Gaussian $q(\mathbf{X})$ and some covariance functions.

Required Expectations

- Need expectations under $q(\mathbf{X})$ of:

$$\log c_i = \frac{1}{2\sigma^2}\left[k_{i,i} - \mathbf{k}_{i,\mathbf{u}}^\top \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{k}_{i,\mathbf{u}}\right]$$

  and

$$\log \mathcal{N}\left(\mathbf{y}|\left\langle\mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{Y})}, \sigma^2\mathbf{I}\right) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\left(y_i - \mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{u}\right)^2$$

- This requires the expectations

$$\left\langle\mathbf{K}_{\mathbf{f},\mathbf{u}}\right\rangle_{q(\mathbf{X})}$$

  and

$$\left\langle\mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{K}_{\mathbf{u},\mathbf{f}}\right\rangle_{q(\mathbf{X})}$$

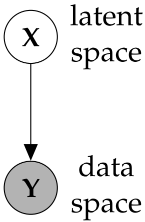
  which can be computed analytically for some covariance functions.

Priors for Latent Space

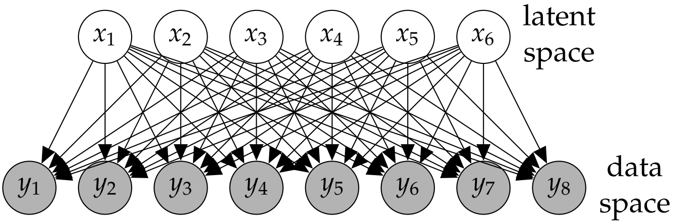
Titsias and Lawrence (2010)

- Variational marginalization of $\mathbf{X}$ allows us to learn parameters of $p(\mathbf{X})$.
- Standard GP-LVM where $\mathbf{X}$ learnt by MAP, this is not possible (see e.g. Wang et al., 2008).
- First example: learn the dimensionality of latent space.

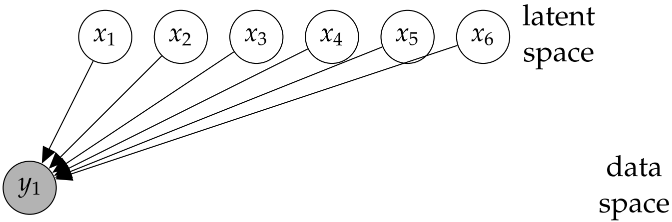
Graphical Representations of GP-LVM



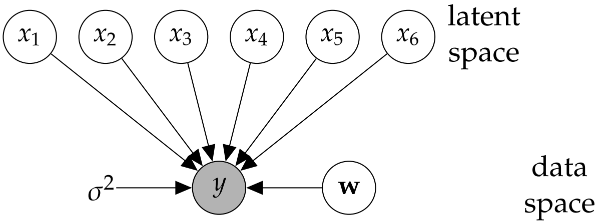
Graphical Representations of GP-LVM



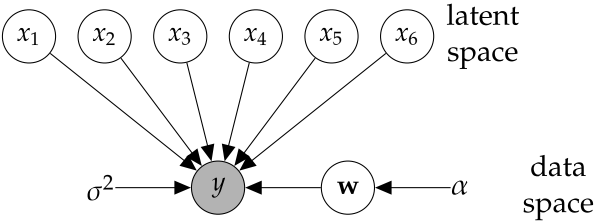
Graphical Representations of GP-LVM



Graphical Representations of GP-LVM



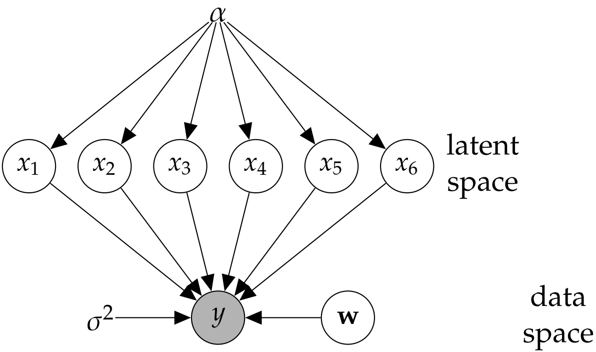
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \alpha\mathbf{I}) \quad \mathbf{x} \sim \mathcal{N}(0, \mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad \mathbf{x} \sim \mathcal{N}(0, \alpha\mathbf{I})$$

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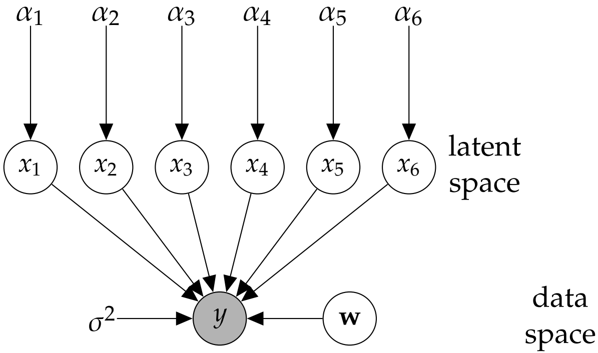
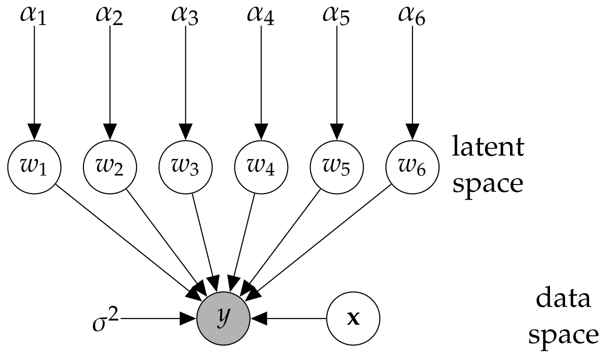
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \qquad x_i \sim \mathcal{N}(0, \alpha_i)$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Graphical Representations of GP-LVM



$$w_i \sim \mathcal{N}(0, \alpha_i) \quad \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Non-linear $f(\mathbf{x})$

- In linear case equivalence because $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$

$$p(w_i) \sim \mathcal{N}(\mathbf{0}, \alpha_i)$$

- In non linear case, need to scale columns of $\mathbf{X}$ in prior for $f(\mathbf{x})$.
- This implies scaling columns of $\mathbf{X}$ in covariance function

$$k(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}) = \exp\left(-\frac{1}{2}(\mathbf{x}_{:,i} - \mathbf{x}_{:,j})^\top \mathbf{A}(\mathbf{x}_{:,i} - \mathbf{x}_{:,j})\right)$$

  $\mathbf{A}$ is diagonal with elements α_i^2. Now keep prior spherical

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \mathbf{I}\right)$$

- Covariance functions of this type are known as ARD (see e.g. Neal, 1996; MacKay, 2003; Rasmussen and Williams, 2006).

Other Priors on $\mathbf{X}$

- Dynamical prior gives us Gaussian process dynamical system (Wang et al., 2006; Damianou et al., 2011)
- Structured learning prior gives us (soft) manifold sharing (Shon et al., 2006; Navaratnam et al., 2007; Ek et al., 2008b,a; Damianou et al., 2012)
- Gaussian process prior gives us Deep Gaussian Processes (Lawrence and Moore, 2007; Damianou and Lawrence, 2013)

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